Math Meth Oper Res (1999) 49:267–282

Mathematical Methods
of Operations Research

Two-person bilateral many-rounds poker

Vladimir V. Mazalov[1], **Svetlana V. Panova**[2], **Mojca Piškurič**[3]

[1] Department of Mathematics and Data Analysis, Karelia Scientific Center, Russian Academy of Sciences, Puskinskaya str. 11, Petrozavodsk 185610, Russia (e-mail: vmazalov@www.krc.karelia.ru)
[2] Department of Mathematics, Chita Pedagogical University, Babushkina str. 129, Chita 672045, Russia
[3] Technical University of Dresden, Institute of Mathematical Stochastics, D-01062, Dresden, Germany

Abstract. This paper analyses a game-theoretic model of Hi-Lo Poker. Bilateral-move N-round games are formulated and explicit solutions are derived. In the asymptotic case the form of optimal decision rule is derived and examples are provided.

Key words: Poker game, optimal strategy, game value

1 Introduction

We study the following zero-sum extensive game. First a move of chance determines the values of variables x and y, which are uniformly and independently distributed in [0, 1]. Player I is informed of the value x and player II is informed of the value y. Subsequently, the players move alternately. On his turn, a player may either *bet* or *pass*. If the first two moves of the game are *pass*, then the game ends; player with the lower hand wins and gets unity from the opponent. Otherwise, the game ends when a player chooses *pass*; if it ends in period t then a player with the higher hand wins and gets the value A_{t-1}, where $\{A_i, i = 1, 2, \dots\}$ is a sequence of real numbers with $1 \le A_1 < A_2 < \cdots$.

A detailed description and discussion on mathematical model of two-person poker is given in Karlin's book [6]. This version of poker relates to the models of Hi-Lo poker (see Sakaguchi [7, 8], Sakaguchi and Mazalov [10]) and preference (Mazalov [9]) with many rounds. In a model of Karlin [6, section 9.5] there is an additional possibility of folding. Our model differs from the model of Karlin without possibility of folding that in case *pass*-*pass* the winner is a player with lower card. Moreover in this model the numbers $\{A_i, i = 1, 2, \dots\}$ may be arbitrary.

Manuscript received: January 1998/final version received: July 1998

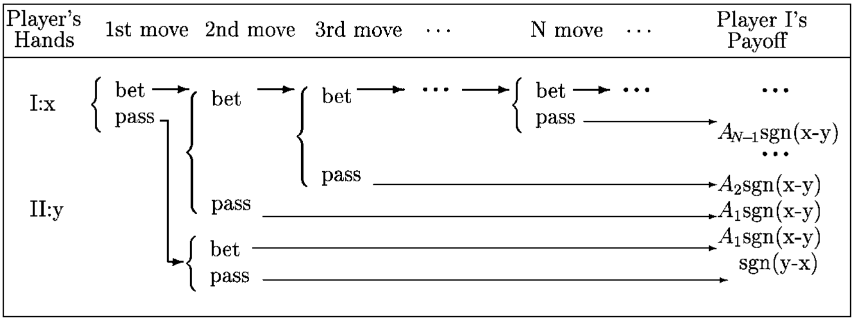
Notice also, that there is a special case of this game with two rounds where both players make decisions simultaneously and $A_2 = 1$ and $A_1 = 1 - 2p$, $p \in [0, 1/2]$. This variant has name of the simple exchange game (see Brams, Kilgour and Davies [1], Garnaev [2, 3], Sakaguchi [4, 5]).

2 Strategies and payoff

The poker discussed here is played by bilateral moves of the players. Let us introduce strategies in the following way. Player I moves first and *bets* with probability α_0, and *passes* with probability $\bar{\alpha}_0 = 1 - \alpha_0$. If z is a probability we denote here $\bar{z} = 1 - z$. Then player II is on the move. If player I on his move made a bet, player II bets with probability β_0 and passes with probability $\bar{\beta}_0$; otherwise, player II bets with probability γ_0 and passes with probability $\bar{\gamma}_0$. If the choice was *pass-pass*, the player with the lower hand wins and gets unity from the opponent.

If both players made a bet player I has the opportunity to correct his previous decision by betting or passing again with probabilities α_1 and $\bar{\alpha}_1$, respectively. If he chooses bet, the second player makes a decision with probabilities β_1, $\bar{\beta}_1$, etc. The process continues until one of the players says *pass*.

Then the players make showdown. Player with the higher hand wins and gets the value A_{N-1} if the round was finished on N-th step. Thus the payoff table is described by:



Notice that the movement may be continued many times. At the beginning we consider the problem with finite horizon N. Later we shall analyse the asymptotic variant. If in the finite case nobody says "pass" we break off the process and suppose that the payoff equals A_N.

3 Case $N = 1$ and $N = 2$

The decision of this game is very simple for $N = 1$. In this case player II passes on second step and the expected payoff of player I be equal

$$M(\alpha_0) = E_{x,y}[\alpha_0(x) A_1 sgn(x - y) + \bar{\alpha}_0(x) sgn(y - x)]$$

$$= (A_1 + 1) \int_0^1 (2x - 1)\alpha_0(x)\, dx.$$

From the form of the payoff it follows that optimal strategy of Player I has form

$$\alpha_0^*(x) = \begin{cases} 0; & \text{if } 0 \le x < 1/2 \\ 1; & \text{if } 1/2 \le x \le 1, \end{cases}$$

and the value of game is equal $V = (A_1 + 1)/4$.

Case $N = 2$ was investigated in details in the article Sakaguchi, Mazalov [10].

Theorem 1 [10]. *The optimal strategies for the game are:*

$$\alpha_0^*(x) = \begin{cases} 0, & \textit{if } 0 \le x < b_0 \\ \textit{arbitrary, but satisfies the requirements} & \\ 0 \le \alpha_0^*(x) \le 1 \quad \textit{and} \quad \int_{b_0}^{b_1} \alpha_0^*(x)\,dx = 1/2 - b_0, & \textit{if } b_0 \le x \le b_1 \\ 1, & \textit{if } b_1 < x \le 1, \end{cases}$$

$$\beta_0^*(y) = I_{(y \ge b_1)} \quad \textit{and} \quad \gamma_0^*(y) = I_{(y \ge b_0)},$$

where

$$b_0 = (A_2 - A_1)/2(A_2 + 1) \quad \textit{and} \quad b_1 = b_0 + 1/2$$

and I_A is indicator of A.

The value of the game is $-(1/4)(A_2 - A_1)(A_1 + 1)/(A_2 + 1)$.

Remark 1. We see that the value of the game is positive for $N = 1$ and negative for $N = 2$.

4 Case $N = 3$

Let us continue the considerations in the case with 3 moves. In this case the expected payoff to I under the strategy collection $(\alpha_0(x), \alpha_1(x), \beta_0(y), \gamma_0(y))$ will be

$$M(\alpha_0, \alpha_1, \beta_0, \gamma_0) = E_{x,y}[\{\alpha_0(x)\beta_0(y)(A_3\alpha_1(x) + A_2\bar{\alpha}_1(x)) + A_1\alpha_0(x)\bar{\beta}_0(y) + \bar{\alpha}_0(x)(A_1\gamma_0(y) - \bar{\gamma}_0(y))\}sgn(x - y)]. \tag{1}$$

Our task is to find player's strategies $\alpha_0^*(\cdot), \alpha_1^*(\cdot)$ and $\beta_0^*(\cdot), \gamma_0^*(\cdot)$ that constitute the saddle point of this function. We shall prove the following result.

Theorem 2. *Optimal strategies for the game with payoff function (1) are*

$$\alpha_0^*(x) = \begin{cases} 0; & \textit{if } 0 \le x < b_0 \\ \textit{arbitrary, but satisfies the requirements} & \\ 0 \le \alpha_0^*(x) \le 1 \quad \textit{and} \quad \int_{b_0}^{b_1} \alpha_0^*(x)\,dx = b_1 - 2b_0; & \textit{if } b_0 \le x < b_1 \\ 1; & \textit{if } b_1 \le x \le 1 \end{cases}$$

$$\alpha_1^*(x) = I_{(x \ge b_2)} \quad \textit{and} \quad \beta_0^*(y) = I_{(y \ge b_1)}, \gamma_0^*(y) = I_{(y \ge b_0)}$$

where $b_0 < b_1 < b_2$ *and satisfy the relations*

$$\begin{aligned} (A_2 - A_1)\bar{b}_1 &= (A_1 + 1)b_0 \\ b_2 - b_1 + \frac{A_3 - A_1}{A_2 - A_1}\bar{b}_2 &= b_1 - 2b_0 \\ b_2 &= \frac{(1 + b_1)}{2}. \end{aligned} \tag{2}$$

The value of the game is

$$V = -\frac{b_0(A_1 + 1)}{2}.$$

Proof. From the side of player II, (1) can be rewritten as

$$\begin{aligned} M(\alpha_0, \alpha_1, \beta_0, \gamma_0) &= E_y[\beta_0(y)L_1(y|\alpha_0, \alpha_1)] + \text{terms independent of } \beta_0(\cdot) \\ &= E_y[\gamma_0(y)L_0(y|\alpha_0, \alpha_1)] + \text{terms independent of } \gamma_0(\cdot) \end{aligned}$$

where

$$L_1(y|\alpha_0, \alpha_1) = \left[-\int_0^y + \int_y^1\right]\alpha_0(x)(A_3\alpha_1(x) + A_2\bar{\alpha}_1(x) - A_1)\,dx$$

and

$$L_0(y|\alpha_0, \alpha_1) = \left[-\int_0^y + \int_y^1\right](A_1 + 1)\bar{\alpha}_0(x)\,dx$$

If player II wants to minimize his loss, i.e. wishes to minimize the payoff function, given an arbitrary strategy pair $(\alpha_0(\cdot), \alpha_1(\cdot))$, he has to choose his optimal strategies in such manner that for all of them the following will be true

$$\text{strategy}_i^*(y) = \begin{cases} 0; & \text{if } L_i > 0 \\ \text{arbitrary}; & \text{if } L_i = 0 \\ 1; & \text{if } L_i < 0 \end{cases}$$

It can easily be verified that both L_1 and L_0 are non-increasing curves satisfying the relation

$$L_i(0|\alpha_0, \alpha_1) > 0 > L_i(1|\alpha_0, \alpha_1) \quad i = 0, 1$$

except in the cases $\alpha_0(x) = 1$ and $\alpha_0(x) = 0$. Consequently if we determine b_0 and b_1 such that

$$L_i(b_i|\alpha_0, \alpha_1) = 0 \quad i = 0, 1,$$

i.e.

$$\begin{aligned} &\left[-\int_0^{b_1} + \int_{b_1}^1\right](A_3\alpha_1(x) + A_2\bar{\alpha}_1(x) - A_1)\alpha_0(x)\,dx = 0 \\ &\qquad\qquad\qquad \left[-\int_0^{b_0} + \int_{b_0}^1\right]\bar{\alpha}_0(x)\,dx = 1 - 2b_0 \end{aligned} \tag{3}$$

we see that the optimal response to I's strategy $(\alpha_0(\,\cdot\,), \alpha_1(\,\cdot\,))$ is

$$\beta^*(y) = I(y \geq b_1) \quad \text{and} \quad \gamma_0^*(y) = I(y \geq b_0).$$

Notice that if $\alpha_0(x), \alpha_1(x)$ have the same form as in Theorem the system (3) is equivalent to

$$\begin{aligned} &\int_{b_0}^{b_1} \alpha_0(x)\,dx = b_1 - 2b_0 \\ &\int_{b_0}^{b_1} \alpha_0(x)\,dx = b_2 - b_1 + \frac{A_3 - A_1}{A_2 - A_1}\bar{b}_2. \end{aligned} \tag{4}$$

which is true in view of (2).

On the other hand – from the side of player I, (1) can be rewritten as

$$\begin{aligned} M(\alpha_0, \alpha_1, \beta_0, \gamma_0) &= E_x[\alpha_0(x)K_0(x|\alpha_1, \beta_0, \gamma_0)] + \text{terms independent of } \alpha_0(\,\cdot\,) \\ &= E_x[\alpha_1(x)K_1(x|\alpha_0, \beta_0, \gamma_0)] + \text{terms independent of } \alpha_1(\,\cdot\,) \end{aligned}$$

where

$$\begin{aligned} K_0(x|\alpha_1, \beta_0, \gamma_0) = \left[\int_0^x - \int_x^1\right](&\alpha_1(x)(A_3 - A_2)\beta_0(y) + A_2\beta_0(y) + A_1\overline{\beta}_0(y) \\ &- A_1\gamma_0(y) + \overline{\gamma}_0(y))\,dy \end{aligned} \tag{5}$$

and

$$K_1(x|\alpha_0,\beta_0,\gamma_0) = \alpha_0(x)\left[\int_0^x - \int_x^1\right](A_3 - A_2)\beta_0(y)\,dy.$$

Player I would of course like to maximize his expected gain from the game, given an arbitrary set of strategies of the second player. Before we determined two functions – strategies of player II – which minimize the payoff function from his side. Now we shall temporally assume that $\beta_0^*(y) = I(y \geq b_1)$, $\gamma_0^*(y) = I(y \geq b_0)$ with b_0, b_1 satisfying relations (2) and $0 \leq b_0 < b_1 \leq 1$ and try to maximize the payoff function. Player II's optimal strategies, together with (4) give us

$$K_0(x|\alpha_1,\beta_0^*,\gamma_0^*) = \begin{cases} 2(A_1+1)x - (A_2 - A_1)\bar{b}_1 \\ \quad - (A_1+1)b_0 - \alpha_1(x)(A_3 - A_2)\bar{b}_1; & \text{if } x < b_0 \\ -(A_2 - A_1)\bar{b}_1 + (A_1+1)b_0 \\ \quad - \alpha_1(x)(A_3 - A_2)\bar{b}_1; & \text{if } b_0 \leq x < b_1 \\ 2(A_2 - A_1)x - (A_2 - A_1)(b_1 + 1) \\ \quad + (A_1+1)b_0 \\ \quad + \alpha_1(x)(A_3 - A_2)(2x - b_1 - 1); & \text{if } x \geq b_1 \end{cases} \tag{6}$$

and

$$K_1(x|\alpha_0,\beta_0^*,\gamma_0^*) = \alpha_0(x)(A_3 - A_2)\begin{cases} b_1 - 1; & \text{if } x < b_1 \\ 2x - b_1 - 1; & \text{otherwise.} \end{cases}$$

From the form of the function $K_1(x|\alpha_0,\beta_0^*,\gamma_0^*)$ we see that independently of $\alpha_0(x)$ the optimal response of player I is

$$\alpha_1^*(x) = I\left(x \geq \frac{b_1+1}{2}\right).$$

Now we can rewrite (6) as

$$K_0(x|\beta^*,\gamma^*) = \begin{cases} (A_1+1)(2x - b_0) - (A_2 - A_1)\bar{b}_1; & \text{if } x < b_0 \\ -(A_2 - A_1)\bar{b}_1 + (A_1+1)b_0; & \text{if } b_0 \leq x < b_1 \\ (A_2 - A_1)(2x - b_1 - 1) + (A_1+1)b_0; & \text{if } b_1 \leq x < \frac{b_1+1}{2} \\ (A_3 - A_1)(2x - b_1 - 1) + (A_1+1)b_0; & \text{if } x \geq \frac{b_1+1}{2}. \end{cases}$$

The form of the function $K_0(x)$ is shown in Figure 1.

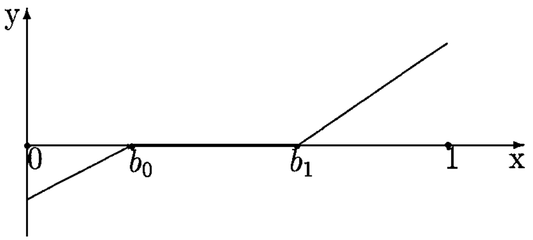


Fig. 1.

The condition

$$K_0(b_0|\beta^*, \gamma^*) = 0, \quad i = 0, 1,$$

i.e.

$$(A_2 - A_1)\bar{b}_1 = (A_1 + 1)b_0$$

follows from (2). Therefore the optimal strategy α_0^* is

$$\alpha_0^*(x) = \begin{cases} 0; & \text{if } x < b_0 \\ \text{arbitrary}; & \text{if } b_0 \le x < b_1 \\ 1; & \text{if } x \ge b_1. \end{cases}$$

Finally we have to compute the value of the game. From (1) we obtain

$$M(\alpha_0^*, \alpha_1^*, \beta_0^*, \gamma_0^*) = \int_{b_1}^1 K_0(x)\,dx + \int_0^1 (1 - 2y)(A_1\gamma_0^*(y) - \bar{\gamma}_0^*(y))\,dy.$$

After simplification we have as the value of the game

$$V = -(A_1 + 1)b_0/2,$$

which completes the proof of the theorem.

5 Finite horizon case

The aim of this section is to prove a general result for any fixed N. For conveniency we suppose that $N = 2n + 2$, i.e. that player II makes the final move (case $N = 2n + 1$ is completely analogous).

Hence the expected payoff of player I takes the form

$$M(\alpha,\beta,\gamma_0) = E_{x,y}[(\alpha_0(x)\{\beta_0(y)\{\alpha_1(x)\cdots[\beta_{n-1}(y)\{\alpha_n(x)\{\beta_n(y)A_{2n+2} + \bar\beta_n(y)A_{2n+1}\} + \bar\alpha_n(x)A_{2n}\} + \bar\beta_{n-1}(y)A_{2n-1}] + \cdots + \bar\alpha_1(x)A_2\} + \bar\beta_0(y)A_1\} + \bar\alpha_0(x)\{A_1\gamma_0(y) - \bar\gamma_0(y)\})sgn(x-y)]. \quad (7)$$

Theorem 3. *The optimal strategies in the game with payoff function (7) take the form*

$$\alpha_0(x) = \begin{cases} 0, & \text{if } 0 \le x \le b_0, \\ arbitrary, & \text{if } b_0 < x < b_1, \\ 1, & \text{if } b_1 \le x \le 1, \end{cases} \quad \alpha_i(x) = I_{(x \ge b_{2i})}, i = 1,\dots,n, \quad (8)$$

$$\gamma_0(y) = I_{(y \ge b_0)}, \quad \beta_i(y) = I_{(y \ge b_{2i+1})}, i = 0,\dots,n, \quad (9)$$

with $b_i, i = 0,\dots,2n+1$, *satisfying the system of equations*

$$\begin{cases} b_{2n+1} = \dfrac{1+b_{2n}}{2} \\ (A_{2n+2} - A_{2n+1})(1-b_{2n+1}) = (A_{2n+1} - A_{2n})(2b_{2n} - b_{2n-1} - 1) \\ \vdots \\ (A_{2i+2} - A_{2i+1})(1-b_{2i+1}) = (A_{2i+1} - A_{2i})(2b_{2i} - b_{2i-1} - 1) \\ \vdots \\ (A_4 - A_3)(1-b_3) = (A_3 - A_2)(2b_2 - b_1 - 1) \\ (A_3 - A_2)(1-b_2) = (A_2 - A_1)(2b_1 - 2b_0 - 1) \\ (A_2 - A_1)(1-b_1) = (A_1 + 1)b_0 \end{cases} \quad (10)$$

and

$$\int_{b_0}^{b_1} \alpha_0(x)\,dx = b_1 - 2b_0. \quad (11)$$

The value of the game is $V = -\frac{A_1+1}{2}b_0$.

Proof. Before we prove theorem let us find the solution of system (10).

Lemma. *The solution of system (10) exists and satisfies the relation:*

$$0 < 2b_0 < b_1 < b_2 < \cdots < b_{2n+1}. \quad (12)$$

The *Proof* of the above lemma follows from the presentation of the equations in system (10) in recurrent form $b_i = x_i + y_i b_{i-1}$.

$$\begin{cases} b_{2n+1} = \dfrac{1}{2} + \dfrac{1}{2} b_{2n} \\ b_{2n} = \dfrac{\frac{1}{2}\Delta_{2n+1} + \Delta_{2n}}{2\Delta_{2n} + \frac{1}{2}\Delta_{2n+1}} + \dfrac{\Delta_{2n}}{2\Delta_{2n} + \frac{1}{2}\Delta_{2n+1}} b_{2n-1} \\ \vdots \\ b_i = x_i + y_i b_{i-1} \\ b_{i-1} = \dfrac{(1-x_i)\Delta_i + \Delta_{i-1}}{2\Delta_{i-1} + \Delta_i y_i} + \dfrac{\Delta_{i-1}}{2\Delta_{i-1} + \Delta_i y_i} b_{i-2} \\ \vdots \\ b_2 = \dfrac{(1-x_3)\Delta_3 + \Delta_2}{2\Delta_2 + \Delta_3 y_3} + \dfrac{\Delta_2}{2\Delta_2 + \Delta_3 y_3} b_1 \\ b_1 = \dfrac{(1-x_2)\Delta_2 + \Delta_1}{2\Delta_1 + \Delta_2 y_2} + \dfrac{2\Delta_1}{2\Delta_1 + \Delta_2 y_2} b_0 \\ b_0 = \dfrac{(1-x_1)\Delta_1}{\Delta_0 + \Delta_1 y_1} \end{cases} \tag{13}$$

where we have used the notation $\Delta_i = A_{i+1} - A_i, \Delta_0 = A_1 + 1$.

y_i determine all b_i uniquely, and for y_i we have the following recurrent formulas

$$\begin{cases} y_i = \dfrac{\Delta_{i-1}}{\Delta_i}\left(\dfrac{1}{y_{i-1}} - 2\right), i = 3, \ldots, 2n+1, \\ y_2 = \dfrac{\Delta_1}{\Delta_2}\left(\dfrac{2}{y_1} - 2\right), \end{cases}$$

and

$$y_1 = \cfrac{2\Delta_1}{2\Delta_1 + \cfrac{\Delta_2^2}{2\Delta_2 + \cfrac{\Delta_3^2}{2\Delta_3 + \ddots + \cfrac{\Delta_{2n}^2}{2\Delta_{2n} + \frac{1}{2}\Delta_{2n+1}}}}} \tag{14}$$

from where we deduce the existence and uniqueness of the solutions of system (10).

Notice that geometrically the relations

$$b_i = x_i + y_i b_{i-1}, \quad x_i + y_i = 1, \quad x_i > 0, \quad y_i > 0, \quad i = 2, \ldots, 2n+1,$$

mean that the value b_i lies between b_{i-1} and 1. Therefore $0 < b_1 < b_2 < \cdots < b_{2n+1} < 1$.

From (13) it follows that for $i = 1$ $b_1 = x_1 + (1 - x_1)2b_0$. Hence b_1 lies between 1 and $2b_0$, i.e. $0 < 2b_0 < b_1$. Thus we have proven the inequalities (12).

Now let us prove the theorem. First consider the problem from the side of player II. Let us suppose that player I uses strategies of the form (8) with b_i satisfying (10) and try to find the optimal response of player II. His aim is to minimize the payoff (7).

Let us rewrite (7) in the form

$$M(\alpha, \beta, \gamma) = \int_0^1 \gamma_0(y) L_0(y)\, dy + \sum_{i=0}^{n} \int_0^1 \prod_{j=0}^{i} \beta_j(y) L_i(y)\, dy, \tag{15}$$

where

$$L_0(y) = \int_0^1 (A_1 + 1)\bar{\alpha}_0(x) sgn(x - y)\, dx = (A_1 + 1)\left[-\int_0^y + \int_y^1\right] \bar{\alpha}_0(x)\, dx$$

$$L_n(y) = (A_{2n+2} - A_{2n+1})\left[-\int_0^y + \int_y^1\right] \prod_0^n \alpha_j(x)\, dx,$$

$$L_i(y) = (A_{2i+3} - A_{2i+2})\left[-\int_0^y + \int_y^1\right] \prod_{j=0}^{i+1} \alpha_j(x)\, dx$$

$$+ (A_{2i+2} - A_{2i+1})\left[-\int_0^y + \int_y^1\right] \prod_{j=0}^{i} \alpha_j(x)\, dx; \quad i = 1, \ldots, n-1. \tag{16}$$

Notice that $L_i'(y) \leq 0$, hence $L_i(y)$ is non-increasing. Because $\prod_{j=0}^{i} \alpha_j(x) = \alpha_i(x)$ it follows from the choice of b_{2i+1} (see (10)) that

$$L_i(b_{2i+1}) = (A_{2i+3} - A_{2i+2})\left[-\int_0^{b_{2i+1}} + \int_{b_{2i+1}}^1\right] \alpha_{i+1}(x)\, dx$$

$$+ (A_{2i+2} - A_{2i+1})\left[-\int_0^{b_{2i+1}} + \int_{b_{2i+1}}^1\right] \alpha_i(x)\, dx$$

$$= \Delta_{2i+2}(1 - b_{2i+2}) + \Delta_{2i+1}(1 - 2b_{2i+1} + b_{2i}) = 0.$$

The graph of $L_i(y)$ is shown in Fig. 2.

From here it follows that the optimal response of Player II is $\beta_i(y) = I_{(y \geq b_{2i+1})}$, $i = 0, \ldots, n$. Analysis of the function $L_0(x)$ shows that $\gamma_0(y) = I_{\{y \geq b_0\}}$. Using the same argument for Player I we can rewrite (7) in the following form

$$M(\alpha, \beta, \gamma) = \sum_{i=1}^{n} \int_0^1 \prod_{j=0}^{i} \alpha_j(x) K_i(x)\, dx + \int_0^1 \alpha_0(x) K_0(x)\, dx$$

where

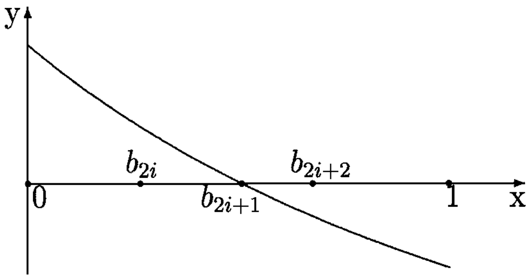


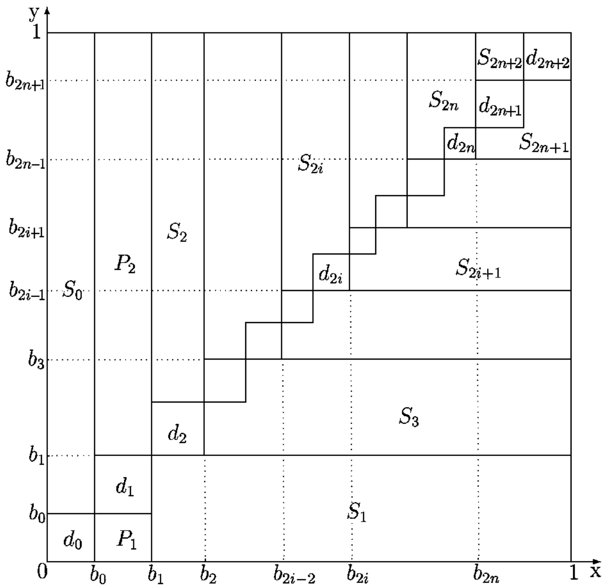
Fig. 2.

$$K_i(x) = (A_{2i+2} - A_{2i+1})\left[\int_0^x - \int_x^1\right]\prod_{j=0}^{i}\beta_j(y)\,dy$$
$$+ (A_{2i+1} - A_{2i})\left[\int_0^x - \int_x^1\right]\prod_{j=0}^{i-1}\beta_j(y)\,dy, \quad i = 1, \ldots, n,$$

and

$$K_0(x) = (A_2 - A_1)\left[\int_0^x - \int_x^1\right]\beta_0(y)\,dy$$
$$- (A_1 + 1)\left[\int_0^x - \int_x^1\right]\gamma_0(y)\,dy + (A_1 + 1)\left[\int_0^x - \int_x^1\right]dy.$$

The functions $K_i(x)$ are increasing and crossing the line O_x in the point b_{2i}. This means that the optimal response of Player I is the collection of strategies (8).



Lastly, it remains to find the value of the game. In the region d_0 the payoff of player I equals $sgn(y-x)$, in regions d_i payoff is $A_i sgn(x-y)$, $i=\overline{1,2n+1}$. Therefore the expectational payoff in these regions equals 0.

Let us calculate the payoff in regions P_1, P_2, S_0, S_1. In the region P_1 the payoff equals $\alpha A_1 - \bar{\alpha}$, in $P_2 - \alpha A_2 - \bar{\alpha} A_1$, in $S_0 - A_1$ and in $S_1 A_1$. Therefore the expectational payoff in these regions is

$$V_0 = b_0(A_1+1)\int_{b_0}^{b_1} \alpha(x)\,dx - b_0(b_1-b_0) - (1-b_1)(A_2-A_1)\int_{b_0}^{b_1} \alpha(x)\,dx$$
$$- A_1(1-b_1)(b_1-b_0) - A_1(1-b_0)b_0 + A_1(1-b_1)b_1.$$

By using (10) we obtain

$$V_0 = [b_0(A_1+1) - (1-b_1)(A_2-A_1)]\int_{b_0}^{b_1} \alpha(x)\,dx - (A_1+1)b_0(b_1-b_0)$$
$$= -(A_1+1)b_0(b_1-b_0).$$

In the region S_i the payoff equals $(-1)^{i-1}A_i$, $i=1,\ldots,2n+1$. Finally we obtain

$$V = M(\alpha^*, \beta^*, \gamma^*) = -(A_1+1)b_0(b_1-b_0) + (A_3-A_2)(1-b_2)(b_2-b_1)$$
$$+ \cdots + (A_{2n-1}-A_{2n-2})(1-b_{2n-2})(b_{2n-2}-b_{2n-3})$$
$$- (A_{2n}-A_{2n-1})(1-b_{2n-1})(b_{2n-1}-b_{2n-2})$$
$$+ (A_{2n+1}-A_{2n})(1-b_{2n})(b_{2n}-b_{2n-1})$$
$$- (A_{2n+2}-A_{2n+1})(1-b_{2n+1})(b_{2n+1}-b_{2n}).$$

and by substituting b_i from the system (10), we get

$$V = -\frac{A_1+1}{2} b_0,$$

which completes the proof of the theorem.

The examples of the optimal strategies and the value of game for $N=4$ and various values of $\{A_i\}$ are given in Table 1.

Remark 2. The value of game is negative. This reflects that Player I stands at a unfavorable condition since he leaks some information about his true to his opponent by moving first. Notice also that he is able to make bluff taking $\alpha_0^*(x)$ arbitrary satisfying condition (11) (for example $\alpha_0^*(x) = (b_1-2b_0)/(b_1-b_0)$). It means that after player I has chosen on first step *bet (pass)*, player II has to guess whether I's hand is truly high (low) and he has made the choice, or I's hand is truly low (high) and he wants to mislead his opponent's choice.

Table 1.

A_i b_i	$A_1 = 2$ $b_0 = 0.10870$	$A_2 = 3$ $b_1 = 0.67391$	$A_3 = 4$ $b_2 = 0.86957$	$A_4 = 5$ $b_3 = 0.93478$	V -0.16304
A_i b_i	$A_1 = 2$ $b_0 = 0.08228$	$A_2 = 3$ $b_1 = 0.75316$	$A_3 = 6$ $b_2 = 0.88608$	$A_4 = 7$ $b_3 = 0.94304$	V -0.12342
A_i b_i	$A_1 = 2$ $b_0 = 0.05093$	$A_2 = 3$ $b_1 = 0.84722$	$A_3 = 11$ $b_2 = 0.92593$	$A_4 = 12$ $b_3 = 0.96296$	V -0.07639
A_i b_i	$A_1 = 2$ $b_0 = 0.24194$	$A_2 = 5$ $b_1 = 0.75806$	$A_3 = 6$ $b_2 = 0.90323$	$A_4 = 7$ $b_3 = 0.95161$	V -0.36290

Remark 3. We see from Table 1 that it is profitable for player I to increase the increment between A_2 and A_3. In this case the value of game tends to zero and the interval of bluffing dilates.

6 The asymptotic case

We constructed above the optimal behavior of players in case of finite horizon. But in real situations the movement may be continued many times without a well-stated maximum bound. It is interesting to analyze the asymptotic behavior of the optimal decisions.

6.1. Uniform increments

Let us suppose here that the increment in the awards in every step has the same value $A_i - A_{i-1} = \Delta$. In this case we obtain the system of equations

$$\begin{cases} \Delta(1 - b_1) = (A_1 + 1)b_0 \\ (1 - b_2) = (2b_1 - 2b_0 - 1) \\ (1 - b_3) = (2b_2 - b_1 - 1) \\ \vdots \\ (1 - b_{2i+1}) = (2b_{2i} - b_{2i-1} - 1) \\ \vdots \\ (1 - b_{2n+1}) = (2b_{2n} - b_{2n-1} - 1) \\ b_{2n+1} = \dfrac{1 + b_{2n}}{2}. \end{cases} \tag{17}$$

Let us consider the case for large N. If $N \to \infty$ and there exists a limit value of $b^* = \lim_{N\to\infty} b_N$ it follows from the last equation in (17) that $b^* = 1$.

To solve the system (17) we can use method of finite differences (Gelfond

Table 2. Case $A_1 = 1, \Delta = 2$

N	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	V
2	0.250	0.750									−0.250
3	0.222	0.778	0.889								−0.222
4	0.227	0.773	0.909	0.955							−0.227
5	0.226	0.774	0.906	0.962	0.981						−0.226
6	0.227	0.773	0.906	0.961	0.984	0.992					−0.227
7	0.227	0.773	0.906	0.961	0.984	0.994	0.997				−0.227
8	0.227	0.773	0.906	0.961	0.984	0.993	0.997	0.999			−0.227
9	0.227	0.773	0.906	0.961	0.984	0.993	0.997	0.999	0.999		−0.227
10	0.227	0.773	0.906	0.961	0.984	0.993	0.997	0.999	1.000	1.000	−0.227
∞	0.227	0.773	0.906	0.961	0.984	0.993	0.997	0.999	1.000	1.000	−0.227

[11]). First we have to solve the uniform equation

$$b_{2i+1} + 2b_{2i} - b_{2i-1} = 0, \quad i = 2, \ldots, 2n.$$

For it we construct the characteristic equation

$$\lambda^2 + 2\lambda - 1 = 0.$$

Its roots are $\lambda_1 = \sqrt{2} - 1, \lambda_2 = -\sqrt{2} - 1$. Consequently

$$b_t = C_1 \lambda_1^t + C_2 \lambda_2^t + 1,$$

and by using the condition $\lim_{t \to \infty} b_t = 1$ we obtain the general solution in the form

$$b_i = 1 + C_1 (\sqrt{2} - 1)^i. \tag{18}$$

The first two equations in (17) say that

$$\begin{cases} (A_1 + 1) b_0 = -\Delta C_1 (\sqrt{2} - 1) \\ -C_1 (3 + 2\sqrt{2}) = 2 C_1 (\sqrt{2} - 1) - 2 b_0 + 1 \end{cases}$$

from where we compute

$$\begin{cases} b_0 = \dfrac{\Delta(\sqrt{2} - 1)}{A_1 + 1 + 2\Delta(\sqrt{2} - 1)} \\[2ex] C_1 = -\dfrac{A_1 + 1}{A_1 + 1 + 2\Delta(\sqrt{2} - 1)}. \end{cases} \tag{19}$$

Below in Table 2 we calculate using (17) the optimal strategies for players and the value of game for various finite N. In final line there are optimal strategies for assymptotic case (see formulas (18)–(19)).

6.2. Geometric increments

Let us consider now the case where the increments appear in the form of a geometric sequence, $A_i = kA_{i-1}, k > 1$. Here we obtain the system of equations

$$\begin{cases} k(1-b_1) = (A_1+1)b_0 \\ k(1-b_2) = (2b_1 - 2b_0 - 1) \\ k(1-b_3) = (2b_2 - b_1 - 1) \\ \vdots \\ k(1-b_{2i+1}) = (2b_{2i} - b_{2i-1} - 1) \\ \vdots \\ k(1-b_{2n+1}) = (2b_{2n} - b_{2n-1} - 1) \\ b_{2n+1} = \dfrac{1+b_{2n}}{2}. \end{cases}$$

and by employing the same arguments as above we get

$$b_i = 1 + C_1\left(\frac{\sqrt{1+k}-1}{k}\right)^i$$

with C_1, b_0 satisfying the equations:

$$\begin{cases} (A_1+1)b_0 = -A_1(k-1)C_1\dfrac{\sqrt{1+k}-1}{k} \\ -kC_1\left(\dfrac{\sqrt{1+k}-1}{k}\right)^2 = 2\left(1 + C_1\dfrac{\sqrt{1+k}-1}{k}\right) - 2b_0 - 1 \end{cases}$$

We have here

$$\begin{cases} b_0 = \dfrac{A_1(k-1)}{(A_1+1)\sqrt{1+k} + 2A_1k - A_1 + 1} \\ C_1 = -\dfrac{(A_1+1)(1+\sqrt{1+k})}{(A_1+1)\sqrt{1+k} + 2A_1k - A_1 + 1} \end{cases}$$

We present in Table 3 the optimal strategies for players and the value of game for various finite N. Compare with assymptotic values in final line.

Remark 4. In table 2 and 3 $A_1 = 1$, consequently, the value of game V coincides with b_0 and negative. We see quick convergence of the decisions and the value of game to limiting values.

Table 3. Case $A_1 = 1, k = 2$

N	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	V
2	0.167	0.667									−0.167
3	0.125	0.750	0.875								−0.125
4	0.136	0.727	0.909	0.955							−0.136
5	0.133	0.733	0.900	0.967	0.983						−0.133
6	0.134	0.732	0.902	0.963	0.988	0.994					−0.134
7	0.134	0.732	0.902	0.964	0.987	0.996	0.998				−0.134
8	0.134	0.732	0.902	0.964	0.987	0.995	0.998	0.999			−0.134
9	0.134	0.732	0.902	0.964	0.987	0.995	0.998	0.999	1.000		−0.134
10	0.134	0.732	0.902	0.964	0.987	0.995	0.998	0.999	1.000	1.000	−0.134
∞	0.134	0.732	0.902	0.964	0.987	0.995	0.998	0.999	1.000	1.000	−0.134

Remark 5. In both cases 4.1 and 4.2 b_0 is an increasing function of Δ and k. Hence the value of game $V = -\frac{(A_1 + 1)}{2} b_0$ is a decreasing function of these parameters. It means that from the side of player I(II) it is profitable to minimize (maximize) the increments between betting values.

Acknowledgements: The authors thank Peter Neumann and Volker Nollau from Dresden Technical University for a kind atmosphere of collaboration and DAAD for providing the funding for scientific stay of VM and MP in Germany. VM and SP were also supported by the Russian Fundamental Research Foundation (project 96-01-01600).

References

[1] Brams SJ, Kilgour DM, Davis MD Unraveling in games of sharing and exchange. to appear
[2] Garnaev A Yu (1992) On simple MIX game. Int. J. Game Th. 21:237–247
[3] Garnaev A Yu (1996) On a MIX game. Game theory and applications, Nova Science Publishers, New York pp. 23–31
[4] Sakaguchi M (1993) On two and three person exchange games. Math. Japonica 38:791–801
[5] Sakaguchi M (1996) Information structures and perfect information in simple exchange games. Game theory and applications, Nova Science Publishers, New York, pp. 168–186
[6] Karlin S (1959) Mathematical methods and theory in games, programming and economics, Vol. II. Pergamon Press, London
[7] Sakaguchi M (1982) Solutions to a class of two-person Hi-Lo poker. Math. Japonica 27:701–714
[8] Sakaguchi M (1993) Information structures and perfect information in some two-person poker. Math. Japonica 38:743–755
[9] Mazalov VV (1996) Game theoretic model of preference. Game theory and applications, Nova Science Publishers, New York pp. 127–137
[10] Sakaguchi M, Mazalov VV (1996) Two person hi-lo poker – stud and draw, I. Math. Japonica 44:39–53
[11] Gelfond AO (1967) Theory of finite differencies. Moscow, Nauka (Russian)