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Computation of Optimal Poker Strategies

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In this paper we discuss aspects of the solution of an eight-person draw poker game. Best strategies against perfect players (game-theoretic strategies) and against lesser players are presented. These strategies generally are functions of pot odds, the positions of relevant opponents, and their playing habits. For example, if player *A* opens with a pair of kings or better (his habit), player *B* should call with a minimum of aces when second from last to act (his position) and the pot equals twice the bet (pot odds are 2 to 1). Strategies of this type are given for opening, calling, and raising. We study the percentage of time one should raise as a bluff before the draw. A simple explanation is given for previously known results concerning optimal bluffing and calling after the draw.

POKER has been studied by, among others, Ankeny [1], Bellman and Blackwell [2], Cover [3], Cutler [4], Friedman [5], Harsanyi [6–8], Karlin and Restrepo [9], Kuhn [10], Nash and Shapley [11], Sion and Wolfe [12], and Von Neumann and Morgenstern [13].

In [14] we present a partial "system" for draw poker. This paper is an attempt to explain how that system was computed.

Section 1 explains how draw poker is played and how poker hands are ranked. A simple method for describing players' positions is given. We present two rules that are assumed to hold throughout the paper. These rules, which are used in many legal poker clubs, allow only limit-sized bets and do not allow checking before the draw. They are important because they simplify the analysis without detracting from its applicability.

Section 2 contains several new tables of basic probabilities that greatly reduce the time required to compute optimal strategies.

In Section 3 optimal bluffing and calling strategy is presented for play against both good and bad opponents. The percentage of time one should bluff and call before the draw is discussed. In particular, we show that one should rarely bluff before the draw against good players. Also, one should bluff less often when the pot is large relative to the bet. A simple explanation is given for previously known results concerning optimal bluffing and calling after the draw.

Section 4 describes how game-theoretic opening strategies may be computed. Essentially, one begins by making an experienced guess as to the opener's best strategy. Let us assume he should open with a pair of kings or better. The optimal calling and raising strategy against someone who opens with a minimum of KK is then computed for each opponent. This information, together with results concerning the effect of betting after the draw and bluffing before the draw, is then used to approximate the opener's expectation with KK. When this expectation is close to zero, we argue that opening with KK is close to the game-theoretic strategy. The optimal opening strategy against weak opponents is also discussed.

Section 5 explains how optimal first-round calling and raising strategies may be determined for play against opponents of varying ability.

1. PRELIMINARIES

For those unfamiliar with poker, we begin by presenting the ranking of poker hands and then discuss the rules of play.

Poker Hands

Poker hands consist of five cards. They are ranked as shown below, best hands first:

1. Straight flush Examples: ◇A◇K◇Q◇J◇10, ♡9♡8♡7♡6♡5.
2. Four of a kind Examples: JJJJ4, 6666K.
3. Full house Examples: QQQ55, 99922.
4. Flush Examples: ◇K◇J◇8◇6◇4, ♡9♡8♡6♡4♡3.
5. Straight Examples: J10987, 65432.
6. Three of a kind Examples: KKK87, 444K6.
7. Two pairs Examples: KK884, 9933K.
8. One pair Examples: QQJ42, 88K64.
9. No pair Examples: AK642, 85432.

The ranking of individual cards in descending order is: A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2. Hands in any category are ranked using a form of lexicographic ordering. For example, because a Q is higher than a 9, QQQ32 beats 999AK, QQ332 beats 9988K, AAQQ2 beats AA99K, QQ432 beats 99AKQ, Q5432 beats 98764, AQ542 beats A9876, etc.

Rules of Play

We will assume a game with 8 players, seated as shown in Figure 1. (The strategies in [14] apply to games with any number of players from 2 to 8.) The first player is numbered 7 because 7 players act after he does. The value of this notation will become clear shortly.

Before any cards are dealt, each player puts a quantity of money, called the *ante*, into the center of the table. The total amount anteed by all the players is called the *total ante*. Thus if each of 8 players antes $1, the total ante is $8.

After the players have anteed, one player shuffles a pack of 52 cards and deals 5 cards face down to each player. The players look at their cards, and the first betting round begins. The player to the dealer's left (player 7) is first to act. He may either bet or check (pass). If he passes, the player to his left has the same option. If all pass, the cards are reshuffled and redealt.

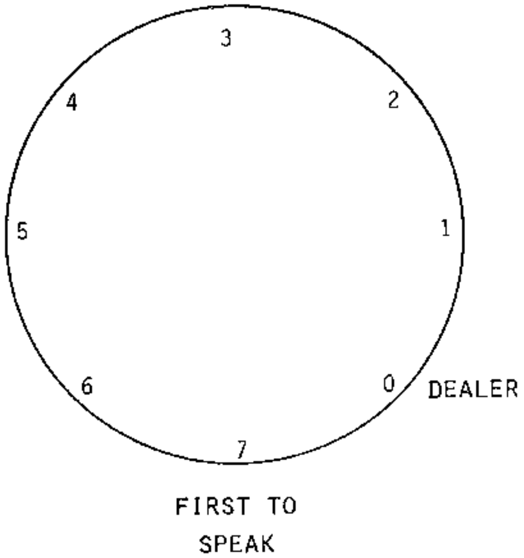

Figure 1. Position before the draw in an 8-handed game.

Once a player has bet (opened), the remaining players have one of three choices. Acting in turn, they may either fold by throwing in their cards, call by putting in as much as the last player who put in chips, or raise by putting in more than that player.

The first betting round ends when either (a) a player makes a bet or raise that is not called by anyone, in which case he wins the pot and a new deal is started; or (b) several players put in equal amounts and everyone else drops, in which case the draw begins.

During the draw each active player may replace any number of cards from his hand for new ones from the unused portion of the deck. After the draw a second betting round begins with rules similar to the first. The active player to the dealer's left is first to speak. A player who checks may later call or raise and win the pot. When all active players check or put in an equal amount of money, they show their hands and the best hand wins the pot. After the winner takes the money from the pot, a new deal is started with the player to the dealer's left becoming the new dealer.

Important Simplifying Assumptions

In recent years poker enthusiasts have embraced a number of rules that speed up the game. Most legal clubs have adopted the following two rules.

1. *Single-size bet rule.* Each bet and raise must equal the limit. For example, if the limit is $10, then each bet and raise must be $10.
2. *No checking on the first round.* Checking is not allowed on the first round. In other words, a player must either bet or drop when it is his turn to open. He cannot check and later come back in to call or raise.

The advantage of rule 1 is that it saves much time normally spent counting chips and making change. When each bet is $10, only one denomination chip is needed to bet or raise. Also, because each bet is the same size, we can and will measure the pot by the number of bets it contains. A pot containing $50 would equal 5 bets.

Rule 2 saves time because it eliminates situations where a player has to act twice before the draw (once to check, once to fold).

I will assume that both rules are in effect because they simplify the analysis considerably and are used in most clubs. The strategies we obtain may be used to get a good feel for what the correct strategy should be when different-sized bets may be made. They also enable one to approximate the correct strategy when checking is allowed.[1]

2. TABLES OF PROBABILITIES

The tables in this section contain probabilities that are used over and over in the computation of optimal strategies.

Table I provides a quick way of determining the probability of a hand without sacrificing accuracy. For example, the exact probability of being dealt a flush equals

$$4\binom{13}{5}/\binom{52}{5} = 5{,}148/2{,}598{,}960 = 0.00198,$$

while the probability that would be read from Table I is 1/520 or 0.00192.

As an example of the use of Table I, suppose we know that the opener

[1] For example, suppose that checking is allowed and players 7 and 6 always check. Then player 5 will have seven active opponents, and consequently should use the same strategy (if he wants to open) that player 7 would use in a game where checking was not allowed. If player 5 always opens when he can, player 4 should use the same strategy as player 6, etc.

TABLE I

CHANCES IN 520 OF BEING DEALT CERTAIN HANDS

Chance	Hand
1	Full house or better
2	Flush or better
4	Straight or better
10	Three eights or better
15	Three deuces or better
21	KK55 or better
27	JJ33 or better
33	8877 or better
40	3322 or better
57	AA or better
74	KK or better
91	QQ or better
261	22 or better

figures to have kings, aces, or something better. Since KK is hand #74, the opener's average hand figures to be hand #37, or roughly two pairs, sixes-up. Therefore, we know that the minimum raising hand should probably be better than 6622.

Table II gives the probability that the best hand dealt to the opponents is $\geqq h$. For example, the probability is 6% that when 7 hands are dealt, the best one among them is $\geqq$ AAA (first row, right-most column). This implies that when we are dealt KKK against 7 other players, the probability is 94% that we have the best hand.

Table II was computed by noting that Pr{player i's hand $\leqq h$|player

TABLE II

DISTRIBUTION OF OPPONENTS' BEST HAND

Hand or better	No. of opponents						
	1	2	3	4	5	6	7
	%	%	%	%	%	%	%
AAA	1	2	3	4	5	5	6*
888	2	4	6	7	9	11	13
222	3	6	8	11	14	16	19
KK77	4	8	12	15	19	22	25
QQ22	5	10	14	19	23	27	31
101022	6	12	17	22	27	31	35
7722	7	14	20	25	30	35	40
3322	8	15	21	27	33	38	43
AA	11	21	29	37	44	50	56

* Probability that the best hand among those dealt to 7 opponents is $\geqq$ AAA

j's hand$\leqq h\} \simeq F_i(h)$, where $F_i(h)$ is the probability that player i is dealt a hand of rank$\leqq h$. In other words, knowing that player j has been dealt less than, say, 101022 has virtually no effect on the probability that player i is dealt less than 101022. To see this, note that there are just as many aces as deuces in the set of all hands three of a kind or better, or in the set of all hands two pairs or better. Therefore, knowing that an opponent's hand is less than 222 or 3322 tells us nothing about the number of high (or low) cards in the remainder of the deck. Of course, if we knew that his hand was less than QQ22, then the remainder of the deck would on the average be infinitesimally richer in queens, kings, and aces. (For example, the expected ratio of aces to total remaining cards would rise from 1/13 to 1/13+0.0002.) Obviously, this would have a negligible effect on the probability of being dealt, say, three of a kind.[2]

Table III gives the probability that one hand will beat another in a showdown, and also the probability of improvement. For example, if player A draws three cards to KK and player B draws three cards to a lower pair, then entry 1 indicates that player B has about a 22% chance of winning in a showdown. The entries in Table III are more general than they might seem, e.g., entry 1 will be very accurate for any pair vs. pair contest in which hand 1>hand 2.[3] Note that two pairs rarely improve (entries 4 and 9).

The following important observation may be made from Table III.

OBSERVATION 1. *A player who calls with a hand that must improve to win is almost always hurting himself and helping opponents unless the ante is very large.*

To see this, suppose that the total ante equals the opening bet. If player A opens with KK and everyone else drops, he wins 1 bet. However, if player B calls with less than KK, player A will have a 78% chance of winning (entry 1), so his expectation, neglecting betting after the draw, will be $3(0.78)-1=1.34$ bets. In other words, player A gains 0.34 of a bet at the expense of his opponent.

As a second example, consider entry 4. In this contest the best hand has

[2] There is one other type of dependency, namely, that holding many cards of one rank, e.g., 8888K or 888KJ, slightly increases the probability that someone else holds many cards of one rank. This type of dependency may also be shown to be insignificant. For example, if player j holds 888KJ, the probability that player i holds exactly three of a kind is changed from 2.3% to 2.5%. If player j holds AA88K, this probability is changed from 2.3% to 2.4%.

[3] Entry 1 was computed by assuming that when hand 1 improves to two pairs, hand 2 must improve to at least three of a kind to win. This assumption will be least accurate when hand 1=33 and hand 2=22. In this case approximately half the time when both hands improve to two pairs hand 2 will win. Using entry 8, the error for this case is approximately $(0.16)^2\,\frac{1}{2}\approx 1.3\%$.

an expectation of $3(0.92)-1=1.76$ bets, and the worst hand has an expectation of -0.76 bets. If another player were to enter the pot with a low two pairs, the situation would be similar to entry 7, in which the best hand has an expectation of $4(0.835)-1=2.34$ bets while the worst hand has an expection of -0.68 bets. Note that the expectations of both players improve when the third player enters the pot.

In practice, one will not know an opponent's exact hand but will often know the worst hand that he opens or calls with. When this information is

TABLE III

A. Probability of Winning in a Showdown

Entry	Hand 1	Hand 2	Hand 3
1	KK, 78%	QQ or any lower pair, 22%	—
2	JJ22, 74%	QQ or any higher pair, 26%	—
3	333, 88%	QQ or any pair above threes, 12%	—
4	KK22, 92%	QQ33 or any lower two pairs, 8%	—
5	AA, 59%	KK, 21%	QQ, 20%
6	JJ22, 54%	AA, 24½%	KK, 21½%
7	KK22, 83½%	QQ33, 8½%	JJ44, 8%

B. Probability of Improving

Entry	Hand	Improvement
8	KK	KKxx, 16%; KKK, 12%; Full house, 1%
9	KK22	Full house, 8½%
10	KKK	Full house, 6%; KKKK, 4%
11	Four-flush	Flush, 19%
12	Four-straight	Straight, 17%

known, Table IV may be used to obtain one's chances of winning against that player. For example, suppose that player A is known to open with a minimum of KK. Player B holds AA. When player A opens, his exact hand will not be known, but entry 9 indicates that player B will have, on the average, a 38% chance of winning in the showdown. This figure was obtained by conditioning on player B's hand as indicated below.

A's Hand	Probability		B's Chances Against	
KK	17/57	×	78%	= 23.3%
AA	3/57	×	50%	= 2.6%
3322-KKQQ	22/57	×	26%	= 10.0%
AAxx	—			
222-KKK	11/57	×	12%	= 2.3%
Straight or better	4/57	×	1%	= 0.1%
				38.3%

The above probabilities were obtained by using Tables I and III plus the observation that the probability of player A having AA or AAxx given that player B has AA is $1/\binom{4}{2}=1/6$ of what it would be ordinarily.[4] Taking the product of the entries on each row and summing gives an answer of 38.3%.

3. CALLING AND BLUFFING STRATEGY

Recall that each bet and raise is assumed to equal the limit, which may be assumed to be one chip. Checking is allowed after the draw but not before. The term "bluff" refers to any bet made with a hand that has virtually no chance of winning in a showdown. A player who bluffs will

TABLE IV
PROBABILITY OF WINNING AGAINST A PLAYER WHOSE WORST LIKELY HAND IS KNOWN

Entry	Hand 1	Opponent's hand	Probability of Hand 1 winning in a showdown
			%
1	222	JJ22 or better	48
2	AA22	JJ22 or better	39½
3	9922	JJ22 or better	8
4	AA	JJ22 or better	16
5	8822	9922 or better	8
6	KK	9922 or better	15½
7	KK	AA or better	19
8	JJ22	KK or better	53
9	AA	KK or better	38

win only when his opponent drops. If a player's chance of winning is $1/(P+1)$, then the *odds* against his winning are P to 1. We say that a player bluffs 1 time for every P legitimate bets if the ratio of his legitimate bets to bluffs is P to 1, or equivalently, if the probability of any particular bet being a bluff is $1/(P+1)$.

We begin with theorems concerning correct play after the draw.

Optimal Calling Strategy After the Draw

Suppose there are two players, A and B, left in the pot after the draw. A bets. B will have no trouble dropping if his hand is very bad and no trouble calling or raising if his hand is very good. Suppose that B has an

[4] The denominator of 57 was obtained as follows: In Table I there are, roughly speaking, 74 hands KK or better, of which 17 are AA and 3 are AAxx. When player B has AA, player A can have $(17/6)\text{AA}\approx 3\text{AA}$ and $(3/6)\text{AA}xx = \frac{1}{2}\text{AA}xx$ or approximately 17 fewer hands, leaving him with roughly 57.

intermediate hand, i.e., one that can beat all bluffs but cannot beat a legitimate bet. Then B will win by calling with this hand only if A is bluffing. Let w denote the probability that A is bluffing, and let $P-1$ denote the number of bets in the pot before A bets. (Note that there are P bets in the pot after A bets.) Then B should call whenever his expectation by calling is positive, i.e., whenever $wP-(1-w)1>0$, or equivalently, whenever $w>1/(P+1)$. This means that B should call whenever A bluffs more than one time for every P legitimate bets.

Suppose now that B does not know how frequently A bluffs. Then if he follows an optimal strategy, it seems reasonable that A's expected gain by bluffing would be zero. Otherwise, A could improve his gain by never bluffing (if gain<0), or by bluffing considerably (if gain>0).

There are exceptions to this observation (see [14, pp. 192–193]), but they are so rare that we will assume it to be true throughout the paper.

Rule 1. When in doubt, a player should call in such a fashion that his opponent will neither gain nor lose in the long run by bluffing.

THEOREM 1. *Fundamental calling theorem.*

Let the pot contain $P-1$ bets after the draw. Two players, A and B remain. A bets. Let w denote the probability that A's bet is a bluff. Then B should call with all intermediate hands if $w>1/(P+1)$ and drop with all intermediate hands if $w<1/(P+1)$. Suppose that B does not know w and that all of his hands can beat a bluff. Then if he drops with probability $1/P$, player A's expected gain by bluffing will be zero.

Proof. The first statement simply says that B should call when his expectation is positive and drop when it is negative. If B drops with probability $1/P$, player A's expected gain by bluffing will be $(1/P)(P-1)-((P-1)/P)1=0$.

Theorem 1 may be extended to multi-player contests. In a contest with m players, a player will break even by bluffing if his opponents all drop collectively with probability $1/P$.

Application 1. A bets after drawing one card to what B believes to be a four-straight or four-flush.[6] B has AA, which is an intermediate hand, i.e., one that will lose to a legitimate bet (a straight or flush) but beat all bluffs. B has observed that A bets quite often (say 1/3 of the time) in these situations. The pot contains 5 bets. What should B do?

Answer: The probability that A makes a straight or flush is about 18% (Table III). If A bets 33% of the time, then the probability of B's winning is 15/33, which is much greater than 1/6. Therefore, B should call.

Application 2. Player A draws three cards. Player B, who raised, draws

[6] If B opened in a good position, he will often be able to infer that A is drawing to a straight or flush since A did not raise and then drew one card.

one card to, presumably, two pairs or three of a kind. A bets, and B does not know what to do. If the pot contained 5 bets after the draw, player B should drop with probability 1/6 (call or raise with probability 5/6). He may as well drop with his worst hands. Therefore, if he raises with a minimum of 9922, he should drop with 9922 through 101099 after the draw, since he will have one of these hands 1/6 of the time. (This may be verified by using Table I and entry 9 of Table III.)

Suppose in the previous application that player A failed to improve and player B bet. Let us assume that player B does not bluff before the draw so that we know he has at least two pairs. In this case player A should not call or raise with probability 5/6 since his hand has no chance of winning. Thus we can state:

Modification of Theorem 1. In general, an uncertain player in a two-player contest after the draw should drop with all hopeless hands and with $1/P$ of the hands that can beat a bluff.

Optimal Bluffing Strategy After the Draw

Optimal bluffing strategy is based on the same principles as the calling strategy. When player A is uncertain as to B's strategy, he should almost always bluff in such a fashion that B will neither gain nor lose by calling with intermediate hands.

THEOREM 2. *Fundamental bluffing theorem.*

Suppose that A has a hand that cannot win if checked. The pot contains $P-1$ bets. Let w denote the probability that B drops after A bets. Then A should bluff if $w>1/P$ and should not bluff if $w<1/P$. Suppose that A does not know w. Then if the ratio of A's bluffs to legitimate bets is $1/P$, B's expected gain by calling with intermediate hands will be zero.

Proof. The first statement simply says that A should bluff when his expectation is positive and check when it is negative. If A's ratio of bluffs to legitimate bets is $1/P$, then B's chances of winning by calling with an intermediate hand will be $1/(P+1)$. His expectation will therefore be $[1/(P+1)](P)-[P/(P+1)](1)=0$.

Theorem 2 says that player A should bluff one time for every P times that he bets legitimately. By doing so, he makes the odds against player B's winning with an intermediate hand equal to player B's pot odds. Theorem 2 has an obvious extension to multi-player contests.

Application. Suppose that in a multi-player contest, player A draws one card to what his opponents easily read to be a four-straight or four-flush. The pot contains 5 bets. Player A fails to make his flush and does not know what to do. How often should he bluff?

Answer: His probability of making a straight or flush is about 0.18

(Table III). He should bluff one time for every 6 legitimate bets, or 18%/6 =3% of the time. In other words, he should bluff 3 times in the 82 times out of 100 that he does not make a hand.

Calling Strategy Before the Draw

In the following discussion we assume that the total ante is one bet, i.e., the total ante equals the opening bet. Each player will be referred to by his position at the table (see Figure 1). Thus, player 6 would be the second player to act in an eight-handed game. (In the previous section we labeled the players alphabetically because we were dealing with play after the draw.)

Suppose that player 7 opens with a pair of kings, players 6 through 1 drop, and player 0 raises. Player 0 should have at least 9922 for a legitimate raise but he could also be bluffing with nothing. If player 0 does have 9922 or better, then player 7 will have a 15% chance of winning (entry 6, Table IV), which means that his expectation after calling, neglecting final round betting, would be $5(0.15)-2=-1.25$ bets. He could drop and simply lose one bet, but if he does this with all hands that figure to lose more than one bet (KK, AA, and 3322 through 101099), he will be dropping about 64% of the time and player 0 will make at least $2(0.64-0.36)=0.56$ bets by bluffing with anything. Player 0's gain by bluffing should be zero.

The following theorem is a precise way of stating how frequently a player must call before and after the draw to make sure that his opponents break even by bluffing. It can be extended to games with several betting rounds like stud.

THEOREM 3. *Calling before the draw.*

Suppose that player j opens and player k raises, either with an absolute bust or with a legitimate raising hand. Let r, c, and d represent the probability that player j legitimately reraises, calls, and drops, respectively. If player j calls, we will assume that he checks after the draw and then calls (or raises) with probability $\bar{c}$. Finally, let p_k denote the probability that player k is called or raised legitimately[6] by someone other than player j, and let P denote the number of bets in the total ante. Then player k's expected gain by raising as a bluff will be zero when

$$(1-p_k)[d(P+1)-2r+c((1-\bar{c})(P+2)-3\bar{c})]-2p_k=0, \qquad (1)$$

or $\bar{c}=\{(1-p_k)[d(P+1)-2r+c(P+2)]-2p_k\}/c(1-p_k)(P+5)$.

Proof. We first claim that player k will lose 2 bets when he is called or

[6] For the sake of simplicity, we assume that no one reraises as a bluff. Such bluffs should be rare.

raised by player l, $l<k$. To see this, suppose that player l calls and the opener drops. Player l's call indicates that his hand is considerably better than the minimum hand that player k would raise with legitimately. For the sake of illustration, suppose that player k raises with a minimum of 9922 and player l calls with a minimum of AA22. To protect himself from being bluffed out by legitimate hands like 9922, 101022, JJ22, QQ22, and KK22, player l must call after the draw in such a fashion that player k's gain by "bluffing" with 9922 through KK22 is zero. In other words, if player k's raise was a bluff, further bluffing will not change his current expectation, which is -2 bets. The same type of argument may be applied if players l or j reraise.

The rest of the proof is simply a matter of expressing player k's expectation. Player k loses 2 bets whenever he is called or raised by player l, $l<k$. This occurs with probability p_k. Otherwise, he wins $P+1$ bets when player j drops, which occurs with probability d, and he loses 2 bets when player j reraises, which occurs with probability r. When player j calls before the draw, player k will win $P+2$ bets with probability $c(1-\bar{c})$ and lose 3 bets with probability $c\cdot\bar{c}$. Equation (1) now follows.

Application. Theorem 3 indicates that a player who raises in a bad position (5 or 6) should be called less frequently by the opener than one who raises in a good position. This is because there is a greater chance that someone else will call or raise (p_k is larger), so the opener does not have to call as much. Also, if the opener calls more before the draw, he can call less afterward.

Suppose that player 7 opens the pot with a minimum of KK and player 4 raises. The total ante equals one bet. Using Table II, assuming that player l, $l<4$ calls with a minimum of AA22, we have $p_4=13\%$. If player 7 drops with KK and with 3322 through 9988, and reraises with 777 or better, then using Table I, we have $r=15\%$, $d=36\%$, and $c=49\%$. Substituting these quantities into Theorem 3 gives $\bar{c}=\{0.87[2(0.36)-2(0.15)+3(0.49)]-2(0.13)\}/(0.49)(0.87)(6)=54\%$. Player 7 can call 54% of the time by calling (or raising) whenever he improves after drawing three to AA, and by calling (or raising) with QQ22 or better after drawing one (or two). This may be verified by using Table I plus entries 8 and 9 of Table III.

Bluffing Before the Draw

By bluffing before the draw, one forces opponents to call with many hands that they would ordinarily drop. In order to bluff, one must generally raise and then either "rap pat" (draw no cards) or draw one or two cards. Since the probability of being dealt a legitimate pat hand is only 1/130, anyone who raps pat more than one or two times in an evening is likely to arouse opponents' suspicions. Against wary opponents, it is often a good idea to raise with a high two pairs or any three of a kind and then rap pat.

Because some legitimate raises will be made with one- and two-card draws, it may also be necessary to make some bluffs with one- and two-card draws. The best hand for the one-card draw bluff is a four-flush. This hand, which ordinarily should be dropped, gives one a 19% chance of winning legitimately. The best hand for a two-card bluff would be either a three-card straight flush like ◇10◇J◇Q, or a high pair with an ace kicker like QQAK10. Note that both hands would ordinarily be dropped.

To call with the intention of bluffing later is generally a bad idea because a call shows weakness and enables other players to enter the pot too easily. The only call-bluff that occasionally works occurs when a player calls in a bad position (6 or 5) and then refuses to draw.

Suppose that player 7 opens, player 4 raises, and player 7 calls. If player 7 follows the strategy given immediately after Theorem 3, it may be verified that player 4 needs at least 222 to bet legitimately after player 7 draws one, and at least 444 to bet legitimately after player 7 draws three. The probability that player 4 can bet legitimately after the draw, given that he raised with a minimum of 9922, is approximately $14/33 \approx 42\%$ (Table I). Since the pot will contain 5 bets[7] after the draw, Theorem 2 implies that player 4 should bluff at most $0.42/6 = 0.07$ times for every legitimate raise. If he bluffed more often, player 7 would always call him after the draw and player 4 would lose by bluffing. Since the probability of player 4 being dealt 9922 or better is $33/520 \approx 6.3\%$, he should bluff at most $(0.063)(0.07) \approx \frac{1}{2}\%$ of the time. This means that after player 7 opens, the probability of someone's raising as a bluff should be less than $7(\frac{1}{2}\%) = 3\frac{1}{2}\%$. Theorem 4 is just a formalization of the above.

Theorem 4. *Bluffing before the draw.*

Suppose that player j opens. Let f_k denote the probability that player k can legitimately raise before the draw and legitimately bet afterward, and let $\bar{P}$ denote the number of bets that are likely to be in the pot after the draw. Then player k should bluff before the draw with probability less than or equal to $f_k/(\bar{P}+1)$.

Proof. The proof is a direct result of Theorem 2.

4. OPENING STRATEGY

The best opening strategy depends on the ratio of the opening bet to the total ante. When the total ante is large in relation to the opening bet (say the total ante is \$20 and the bet \$2), a player can open with weak hands because he is risking little. If the situation were reversed, however

[7] We are assuming that the total ante equals one bet.

(the bet were \$20 and the total ante \$2), then a player would need a very good hand to open. Thus we see

OBSERVATION 2. *As the total ante increases in size relative to the limit, the requirements for opening should decrease.*

We will assume that the total ante equals one bet because this is typically the case in legal clubs. Games with a total ante of two bets are discussed in [14].

We now show how the best opening strategy for player 7 was obtained. (A similar technique was used to determine best strategies for the other players.)

Playing experience quickly convinces one that there is little if any merit in bluffing as the opener. In other words, opening with QQ, KK, AA, etc., is clearly better than opening with 22, KK, AA, etc. Therefore we search for a strategy of the form {open with $h_{\min}$ and with all better hands}. It is natural to assume that $h_{\min}$ will be the worst hand that is profitable to open. This is true, but it must be proved because by opening with $h_{\min}$, one affects the opponents' strategy and thus the gains from opening with all other hands. (The proof is given at the end of the paper.)

We will now show that, in general, a pair of kings is the worst hand that is profitable to open. In particular, it is the worst hand that should be opened against strong opponents.

Assertion. In most games a pair of kings is the worst hand that is profitable to open in position 7 when the total ante equals one bet.

Argument. Let g_7(KK) denote the expectation of player 7 when he opens with a pair of kings. We will begin by approximating g_7(KK) assuming that players 6, 5, ..., 0 use the best calling and raising strategies against someone who opens with kings or better. Then we will approximate g_7(KK) assuming first that (a) they call and raise less frequently than they should and (b) they call and raise more frequently than they should.

Case A. Evaluating g_7(KK) assuming that players 6 through 0 use the best calling and raising strategies.

The best calling and raising strategies against someone who opens with a minimum of kings are given in Table V. The accuracy of these strategies will be justified in the next section. The first row of Table V indicates that player 6 should raise with JJ22 or better and drop otherwise. The bottom row indicates that after player 7 opens and players 6 through 1 drop, player 0 should call with AA through 8877 and raise with 9922 or better.

After player 7 opens with KK, one of three things can happen:

1. Everyone drops.
2. One or more players call, but no one raises.
3. One or more players raise.

In case (1) player 7 wins one bet.

In case (2) it can be shown that player 7's expectation is between -0.2 and -0.3 of a bet.[8] We will take it to be -0.3.

In case (3) player 7's best strategy is to drop,[9] which means that he loses one bet. This loss is incurred whenever player 7 gets raised, even if the raise is a bluff. However, we have seen that player 7 should not be raised as a bluff more than 3.5 % of the time.

In the following we will first assume that all calls and raises are legitimate and then modify the figures obtained to take into account the small chance of a first-round bluff.

TABLE V

MINIMUM LEGITIMATE CALLING AND RAISING HANDS AGAINST ONE OPPONENT WHO OPENS WITH A MINIMUM OF KINGS WHEN THE TOTAL ANTE EQUALS ONE BET

Player	Minimum calling hand	Minimum raising hand
6	JJ22*	JJ22
5	101022	101022
4	8822	9922
3	7722	9922
2	6622	9922
1	AA	9922
0	AA	9922

* Calls can also be made with four card straight-flushes, but these are rare (0.002 probability).

The probability that everyone drops (assuming there are no bluffs) is simply the probability that player 6 receives less than JJ22, player 5 receives less than 101022,..., player 0 receives less than AA. Because of the independence noted earlier, this probability is simply

$$F(101099)F(9988)F(7766)F(6655)F(5544)F(\mathrm{KK})^2 = 0.56$$

where $F(h)$ is the probability that a player is dealt a hand of rank $\leq h$. By a similar argument, it can be shown that the probability that one or

[8] For example, when one opponent calls with aces, player 7 will have a 22% chance of winning (entry 1, Table III); hence his expectation, neglecting final-round betting, will be $(0.22)3-1=-0.34$ bets. When one player calls with two small pairs, player 7's expectation is approximately $0.26(3)-1=-0.22$ bets (entry 2, Table III). If one player calls with aces and another calls with two small pairs, player 7 will have about a 21½% chance of winning (entry 6, Table III); hence his expectation will be approximately $(0.215)4-1=-0.14$ bets, etc. We have not bothered to be extremely accurate here because case 2 does not occur too often (10% of the time). Thus an error of 0.1 in case 2 will account for only an error of 0.01 in the final answer.

[9] For an explanation, see the section on calling before the draw.

more players raise is 0.36. This implies that the probability of case 2 occurring is 0.08.

Thus, player 7's approximate gain by opening with KK against good players, neglecting the possibility of a first-round bluff, is

$$\begin{aligned}
0.56(1) &= 0.56 \text{ (everyone drops)} \\
0.08(-0.03) &= -0.02 \text{ (one or more calls but no raise)} \\
0.36(-1) &= \underline{-0.36} \text{ (one or more raises)} \\
& +0.18 \text{ of a bet.}
\end{aligned} \tag{2}$$

If player 7 is raised as a bluff 3.5% of the time, his gain cannot go down by more than 0.07 of a bet to 0.11 of a bet. In actuality, his gain will be about +0.13 of a bet because some bluffs and legitimate raises will overlap.

Looking closely at (2), we see that player 7 will make money by opening with anything if his opponents follow Table V. This is because the true equilibrium occurs when player 7 opens occasionally with queens, as well as with kings or better. If he does this, his opponents should call slightly more often, making his gain by opening with queens zero. I simply rounded the equilibrium point slightly to avoid unnecessary confusion. If player 7 opens with many weak hands, his opponents will call and raise so frequently that he will lose money on those hands.

Case B. Evaluating g_7(KK) when players 6 through 0 call and raise less often than they should.

It is not difficult to see from (2) that player 7's gain will go up if his opponents call and raise less than they should because the first row of (2) will be more positive and the last row less negative. In this situation player 7 should open as much as possible without making his opponents suspicious, e.g., with queens and possibly with jacks.

Case C. Evaluating g_7(KK) when players 6 through 0 call and raise more often than they should.

Suppose now that players 6 through 0 call with a minimum of aces and that their raising strategy is unchanged. (This is about the worst situation player 7 is likely to encounter in practice.) In this case, using Table II and neglecting the possibility of a first-round bluff, we find that player 7's gain by opening with kings is approximately

$$\begin{aligned}
0.44(1) &= 0.44 \text{ (everyone drops)} \\
0.20(-0.03) &= -0.06 \text{ (one or more calls but no raises)} \\
0.36(-1) &= \underline{-0.36} \text{ (one or more raises)} \\
& +0.02 \text{ of a bet.}
\end{aligned} \tag{3}$$

Player 7 loses about 0.04 of a bet when the possibility of a first-round bluff is considered, which brings his expectation down to −0.02 bets. In other words, it does not matter much whether he opens with kings or not in this case. In summary, it follows that player 7 should generally open with kings or better and should loosen his requirements slightly when his opponents call and raise less often than they should.

Because the calling strategy in Table V is optimal, calling with aces is not in general beneficial for players 6 through 0. The money they gain from this strategy when the opener has specifically kings is more than offset by the money they lose to the opener's good hands like 222 or QQQ99. Thus, while the opener might lose slightly on his kings, his overall gain increases when his opponents adopt such a strategy.

5. CALCULATING OPTIMAL CALLING AND RAISING STRATEGIES

To compute the correct opening strategy for player 7 in the previous section, it was necessary to know the best calling and raising strategies for his opponents. These must be computed beforehand. We will now show how the first row of Table V was computed, i.e., we will show why player 6 should raise with JJ22 or better against good opponents and drop otherwise. (Other entries were computed in a similar fashion.) It suffices to demonstrate that player 6 breaks approximately even by raising with JJ22 and that his gain from calling is less than his gain from raising.

Verification That Player 6 Breaks Approximately Even by Raising with JJ22

After player 6 raises, it may be ascertained that player l, $l<6$, needs approximately 222 to call legitimately and approximately 999 to raise legitimately.[10] Hence, according to Table II, player 6 should be overcalled 5% of the time and reraised legitimately 11% of the time. Since the reraiser needs, on the average, KKK or better to legitimately bet after the draw, and the pot will contain 8 bets if the opener drops, it follows from Theorem 4 and Table II that player 6 should be reraised as a bluff less than $6\%/9 \approx 0.7\%$ of the time.

When player 6 is overcalled, his chances of winning will be about 8% (entry 4, Table III); hence his expectation, assuming that the opener drops, will be approximately $0.08(6)-2=-1.52$ bets.[11] When player 6 is reraised, his best play is to drop. Thus he will lose 2 bets regardless of whether the raise is legitimate or not.

When no one calls or raises, player 6 will be up against the opener. The

[10] The requirement of 222 was determined by using entries 1 and 2 of Table IV, together with the fact that the opener figures to have better than AAKK 20% of the time. Player j will be getting 2 to 1 odds if the opener drops, which means that he would need at least a 33% chance to call. However, his chances with AAKK will be less than 80% (39.5%) = 31.6%.

[11] Player 6's exact expectation, taking into account that the opener might call or raise, is -1.58 bets. We are not bothering to be exceedingly accurate here because the error of 0.06 will be multiplied by 0.05.

opener's best strategy after the raise is to call with everything except KK and 3322 through JJ1010, and to reraise with 999 or better.

When the opener does reraise, player 6's best play will be to drop, losing 2 bets. It may be verified, using an approach similar to that used in Theorem 4, that the opener should not reraise as a bluff more than $1\frac{1}{2}\%$ of the time. (The hands he may bluff with are hands like 9988, which would ordinarily be dropped after a raise.)

From Table I, player 7's chances of having various hands, given that he opened, are as follows: KK—25%, AA—23%, 3322 to 101099—17%, JJ22 to 888—24%, 999 or better—13%. Using entries 2 and 4 of Table III, JJ22 has a 74% chance against AA and an 8% chance against a high two pairs or any three of a kind. Therefore, player 6's gain when he is left against the opener, neglecting the possibility of player 7's reraising as a bluff, is approximately

$$
\begin{array}{lll}
0.23(2) & = 0.46 & \text{(player 7 drops with KK)} \\
0.23(5(0.74)-2) & = 0.39 & \text{(player 7 calls with AA)} \\
0.17(2) & = 0.34 & \text{(player 7 drops with 3322—101099)} \\
0.24(5(0.08)-2) & = -0.38 & \text{(player 7 calls with JJ22—888)} \\
0.13(-2) & = \underline{-0.26} & \text{(player 7 reraises with 999 or better)} \\
 & \, +0.55 & \text{of a bet.}
\end{array}
$$

If player 7 reraises as a bluff $1\frac{1}{2}\%$ of the time with hands that he would ordinarily drop, the above figure becomes 0.49.

Player 6's expectation after raising with JJ22 is therefore approximately

$$
\begin{array}{lll}
0.12(-2) & = -0.24 & \text{(someone reraises, possibly as a bluff)} \\
0.05(-1.52) & = -0.08 & \text{(someone overcalls)} \\
0.83(.49) & = \underline{0.41} & \text{(player 6 is left against the opener)} \\
 & \, +0.09 & \text{of a bet.}
\end{array}
$$

Sensitivity to Variations in Play

Player 6's gain with JJ22 will go down if his opponents overcall and reraise more often than they should; up if they overcall and reraise less often than they should. If they overcall with a minimum of QQ22 and keep the raising strategy the same, then player 6's expectation will go down to about -0.13 bets, which means that he should not raise. (This is about the worst that could happen in practice because player 6 will gain if anyone overcalls with less than JJ22.)

If player 6 continues to raise with JJ22, he will still gain overall when his opponents call with QQ22 because the money he loses with JJ22 will be more than made up for by the additional money he wins on his strong hands like 999.

Most of the errors that might be made by the opener will increase player

6's gain. For example, if the opener were to call with KK and 3322—101099 instead of dropping with these hands, player 6's gain with JJ22 would go up to 0.12 of a bet.

Computation of Player 6's Expected Gain by Calling

We will assume that player 6 calls with 101022 through QQJJ and evaluate his gain from calling with JJ22. Player 6's opponents should raise with QQ22 or better since with QQ22 they are likely to have player 6 beat and have a good raise against the opener. This means that player 6 should get raised legitimately roughly 27% of the time (Table II). Player 6 should almost never be called because his call shows almost exactly JJ22, i.e., his opponents should either raise or fold. The probability of someone's raising as a bluff should be less than 3%. Player 6's best strategy when raised is to drop, losing one bet.

When no one else enters the pot, player 6 will be left against the opener. In this situation, player 6's chance of winning is 53% (entry 8, Table IV); hence his expected gain, neglecting betting after the draw, is $0.53(3)-1=0.59$ of a bet. Because player 6 makes the strength of his hand rather clear by calling and drawing one,[12] the opener should bet with QQ22 or better after the draw. His probability of either starting with or eventually making such a hand is approximately 48% (Table I and entries 8 and 9 of Table III).

When the opener does bet, player 6 will have to call (or raise) approximately three-fourths of the time (Theorem 1). Nine percent of the time he will make a full house and will reraise, picking up approximately $1+2/3$ bets.[13] Player 6's expected loss after the draw is therefore approximately $-0.48(0.66-0.09(1+2/3))\approx-0.25$ of a bet. (We are using the fact that when the opener bluffs, player 6's expected loss is zero [Theorem 1].)

Player 6's total expected gain by calling with JJ22, neglecting the possibility of a first-round raise as a bluff, is therefore approximately

$$
\begin{aligned}
0.27(-1) &= -0.27 \text{ (someone raises)}\\
0.73(0.59-0.25) &= \underline{+0.25} \text{ (alone against the opener)}\\
& \quad -0.02 \text{ of a bet.}
\end{aligned}
$$

When the possibility of a raise as a bluff is considered, the figure drops to about -0.06 of a bet. Thus we see that player 6 makes more by raising with JJ22 than he does by calling. Also, his expectation by raising with JJ22 is closer to zero.

[12] Player 6 could refuse to draw, but this would decrease his chance of winning and when overdone would not prevent the opener from betting.

[13] By raising, player 6 risks 2 bets to win 4 (the total ante, the two first-round bets, and the bet after the draw). Therefore, his opponent should call or raise two-thirds of the time (Theorem 1).

6. FINAL REMARKS ON OPENING

So far we have demonstrated that in most instances player 7 makes money by opening with KK. We will now show that this policy does not significantly affect his overall gain with other hands.

Let $g_7(h|h_{\min}=\text{KK})$ denote player 7's expected gain from opening with hand h, given that he opens with a minimum of KK and his opponents use best play. If player 7 opened with a minimum of aces, his opponents' best strategy would be to either drop or raise, i.e., they should never call.

TABLE VI

MINIMUM LEGITIMATE RAISING HANDS AGAINST ONE OPPONENT WHO OPENS WITH A MINIMUM OF ACES WHEN THE TOTAL ANTE EQUALS ONE BET

Player	Minimum raising hand*
6	KK22
5	QQ22
4	QQ22
3	QQ22
2	QQ22
1	JJ22
0	JJ22

* All weaker hands should be dropped, except four-card straight-flushes, which warrant a call but are rare.

Their best raising strategy is given in Table VI. Comparing Tables VI and V, we see that when player 7 opens with KK, he enables his opponents to raise with JJxx, 1010xx, and 99xx; hands which they would drop if player 7's minimum was AA. Also, he enables opponents in good positions to call with hands like 6622 and 3322. By allowing his opponents to call and raise with medium two pairs, player 7 increases his expected gain on his good hands (QQ22 on up), but decreases his expected gain on his weak hands like AA and 3322, which will now be called and raised more frequently.

The change in total gain that results when player 7 opens with KK may be written as $g_7(\text{KK})f(\text{KK}) + \Delta G$, where

$$\Delta G = \sum_{h \geq \text{AA}} g_7(h|h_{\min} = \text{KK})f(h) - \sum_{h \geq \text{AA}} g_7(h|h_{\min} = \text{AA})f(h),$$

and $f(h)$ is the density function of $F(h)$. We wish to show that $\Delta G \approx 0$. This will be done by conditioning on the opponents' best hand.

Case 1. The opponents' best hand is QQ22 or better.

In this case player 7 will get raised regardless of whether he opens with kings or not, so $\Delta G \approx 0$. However, when $h_{\min} = \text{KK}$, the expected number of opponents in the pot will be very slightly greater. Thus, by Observation 1, ΔG will actually be very slightly positive. (The contribution to ΔG from this case may be shown to be less than 0.001 of a bet.)

Case 2. The opponents' best hand is either JJxx, 1010xx, or 99xx.

We will assume that the opponent's best hand is 1010xx.

Gain when Player 7 Does Not Open with KK

Since the opponents need at least JJ22 to call or raise, they should all drop and player 7 will gain one bet on each of his opening hands. His total gain is $(1-F(\text{KK}))1=0.1096$ of a bet (Table I).

Gain when Player 7 Opens with KK

Let us assume here that one player raises with 1010xx and everyone else drops. This assumption will cause us to slightly underestimate ΔG (Observation 1). Player 7's expected gain with his various hands (assuming everyone plays well) will be as follows:

Hands	Probability	Gain in Bets	Explanation
999 or better	1.8%	3.0	Player 7 reraises, opponent drops.
JJ22—888	3.6%	2.6[14]	Player 7 calls.
1010xx	—		It is unlikely that both players have 1010xx.
3322—9988	2.0%	−1.0	Player 7 drops.
AA	3.4%	−0.7[15]	Player 7 calls.

Multiplying the entries in each row and adding, we find that player 7's overall gain is 0.1038 of a bet. Hence, ΔG in this case equals $0.1038-0.1096 = -0.0058$ of a bet.

Case 3. One opponent calls with a hand that is at least AA but less than 9922.

In this case most of player 7's hands will be better than the hand of the player who calls, and it may be verified using techniques similar to those used in case 2 that $\Delta G \approx 0.038$.

Using Table II, the probability that case 2 occurs is 6%, while the probability that case 3 occurs is approximately 10%. Therefore, ΔG is approximately $(0.10)\,(0.038)-(0.06)\,(0.0058) \approx 0.003$ of a bet. Thus we see that opening with KK has a negligible effect on the gain from opening with other hands.

[14] Player 7 has a 92% chance of winning (entry 4, Table III), so his expectation is roughly $(0.92)5-2=2.6$ bets.

[15] Player 7 has a 26% chance (entry 2, Table III), so his expectation is roughly $0.26(5)-2=-0.7$ of a bet.

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